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ESTIMATE OF THE ERROR OCCURRING IN THE LINEARIZATION OF GEOMETRICALLY NON-LINEAR PROBLEMS OF THE THEORY OF ELASTICITY*

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Error estimates are constructed for solving geometrically linear elasticity theory problems for small displacement gradients. It is shown that the solutions of geometrically non-linear and their corresponding linear problems differ in the norm L_2 by terms of the order of the displacement gradients themselves. Certain sufficient conditions for which the linearization procedure is legitimate for small deformations are established as a function of the geometry of the elastic body in the undeformed state. The reasons why the linearization of the non-linear problem for small deformations can result in an error for bodies of shell or bar type are clarified.

Kirchhoff /1/ himself noted that the identification of the deformed and undeformed states of a body, customary in classical (geometrically linear) elasticity theory, is a supplementary hypothesis of a mathematical nature. Indeed, the problem of geometrically non-linear theory in the case when the displacement gradients are small can be considered as a problem with a small parameter. Ascribing the order of smallness Δ to the displacement gradients and discarding terms of order Δ and higher as compared with the principal terms in all the relationships of the geometrically non-linear theory, we obtain the equations of the geometrically linear theory. It is clear that such a procedure induces an error in the solution of the geometrically non-linear problem. Consequently, it is important to estimate it and thereby to provide a basis for the legitimacy of linearizing geometrically non-linear problems for small displacement gradients.

1. The error of the linear theory for small displacement gradients.

The error estimate of linear-theory solutions will be understood below to be the estimate of the difference $p^\circ - p^*$ in some norm, where p^* is the solution of the geometrically non-linear problem while p° is the solution of its corresponding geometrically linear problem (p°, p^* can be stress, strain, or displacement fields of points of the elastic body). The error of the solutions should not be confused with the error of the linear-theory equations themselves. This latter will be defined as the relative magnitude of the small terms being discarded as compared with the principal terms in the relationships of geometrically non-linear theory and equals Δ.

Let x^i be the Cartesian coordinates of an observer in R^3. Let ξ^a denote the individual Lagrange coordinates of points of an elastic body. Later the indices $s, j, k \ldots, a, b, c \ldots$ take the values 1, 2, 3; the former correspond to projections on the x^i coordinate axes, and the latter, on the axes ξ^a. We denote the coordinates of points of an elastic body in the undeformed state by $x^{oi}(\xi^a)$ and in the deformed state by $x^i(\xi^a)$. In the undeformed state the parameters ξ^a yield a certain curvilinear system of coordinates in R^3 whose metric tensor is $g_{ab}{}^\circ \approx \delta_{ij} x_a{}^{oi} x_b{}^{oj}$; $x_a{}^{oi} \equiv x_{i,a}{}^\circ \equiv \partial dx^{oi}/\partial\xi^a$. Deformations of a continuous medium are described by the tensor $\varepsilon_{ab} = (x_a{}^i x_{i,a} - g_{ab}{}^\circ)/2$, $x_a{}^i \equiv x_{i,a} \equiv \partial x^2/\partial\xi^a$. It is expressed in terms of the displacement vector as follows:

$$\begin{aligned} &\varepsilon_{ab} = (\overset{\circ}{x}{}_a^i w_{i,b} + \overset{\circ}{x}{}_b^i w_{i,a} + w^i_{,a} w_{i,b})/2 \\ &w^i(\xi^1) = x^i(\xi^2) - x^{oi}(\xi^a) \end{aligned} \tag{1.1}$$

Later the indices $a, b, c \ldots$ will be discarded by using the metric $g_{ab}{}^\circ$.

We assume that the elastic body that occupies the domain V in the undeformed state is deformed under the effect of certain "dead" weights F_i and surface forces P_i given on the parts S_σ of the boundary ∂V of the domain V. The true solution of the problem posed is found from the condition of stationarity of the Lagrange functional /2, 3/

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$$I(x^i(\xi^a)) = \int_V U(\varepsilon_{ab})\,d\tau - \int_V F_i x^i\,d\tau - \int_{S_\sigma} P_i x^i\,d\sigma \tag{1.2}$$

in the set of functions $x^i(\xi^a)$ satisfying the constraints

$$x^i(\xi^a)|_{S_u} = \bar{x}^i, \quad S_u = \partial V \setminus S_\sigma \tag{1.3}$$

($d\tau$ is the volume element of the domain V, $d\sigma$ is the element of area of the surface S_σ). Conditions (1.3) mean that displacements of the points are given on part of the boundary of the elastic body S_u.

Furthermore, a physically linear elastic body is considered. Its elastic energy density has the form $U = {}^1/_2\, E^{abcd}\varepsilon_{ab}\varepsilon_{cd}$ (F^{abcd} is the elastic-moduli tensor).

The Euler equations of the variational problem (1.2), (1.3) are written as follows:

$$p_{i;a}^a + F_i = 0, \quad p_i^a = \frac{\partial U}{\partial x_a^i} = x_{i,b}\frac{\partial U}{\partial \varepsilon_{ab}} = \sigma^{ac} x_{ic} \tag{1.4}$$

$$p_i^a n_a|_{S_\sigma} = P_i \tag{1.5}$$

where p_i^a is the Piola-Kirchhoff stress tensor, n_a are components of the normal vector to S_σ, and the semicolon denotes the operation of covariant differentiation with respect to the affine property of the ξ^a coordinate system in the undeformed state. Eqs.(1.4) and (1.5) are a closed system of equations of the geometrically non-linear theory of elasticity for a physically linear material.

The closed system of linear-theory equations contains the boundary conditions (1.3), (1.5), the equilibrium Eq.(1.6), the equation of state connecting the stress tensor σ^{ab} with the strains (1.7), and the linear connection between the strains and stresses (1.8):

$$\sigma_{;b}^{ab} + F^a = 0 \tag{1.6}$$

$$\sigma^{ab} = \partial U / \partial \varepsilon_{ab} \tag{1.7}$$

$$\varepsilon_{ab} = (x_a^{oi} w_{i,b} + x_b^{oi} w_{i,a})/2 \tag{1.8}$$

The technique for obtaining error estimates of the solutions of linear theory for small displacement gradients is based on the Prager-Synge identity /4/. Let us formulate it.

Let σ_{ab}° be the true solution of the linearized problem. Let $\bar{\sigma}_{ab}$ denote the statically admissible stress field in the linear problem, i.e., the stress field satisfying (1.6) and the boundary conditions (1.5). If w^i is the displacement field of points of an elastic body that satisfies the kinematic conditions (1.3), then the stress field σ_{ab} corresponding to the displacement field w^i by means of (1.8) and (1.7) is called kinematically admissible.

The integral

$$E^*(\sigma_{ab}) = \frac{1}{2}\int_V E^{*abcd}\sigma_{ab}\sigma_{cd}\,d\tau$$

where E^{*abca} is the tensor of elastic compliance, determines the additional energy of a geometrically and physically linear elastic body. The identity

$$E^*[\overset{\circ}{\sigma}_{ab} - (\bar{\sigma}_{ab} + \sigma_{ab})/2] = E^*(\bar{\sigma}_{ab} - \sigma_{ab})/4 \tag{1.9}$$

has been proved /4/.

Since the additional energy is a positive-definite quadratic form in σ_{ab}, then $E^{*1/2}$ can be identified with the norm $L_2(V)$ in the space of all possible stress fields:

$$\alpha \|\sigma_{ab}\|^2 \leqslant E^*(\sigma_{ab}) \leqslant \gamma \|\sigma_{ab}\|, \quad \|\sigma_{ab}\| = \Big[\int_V \sigma^{ab}\sigma_{ab}\,d\tau\Big]^{1/2} \tag{1.10}$$

The constants α and γ depend only on the elastic compliance tensor E^{*abcd}.

The identity (1.9) shows that the stress field $(\bar{\sigma}_{ab} + \sigma_{ab})/2$ approaches the exact solution of the problem in the norm L_2 well, provided the additional energy is small in the difference between the statically and kinematically admissible stress fields. In solving the non-linear problem σ_{ab}^* the fields $\bar{\sigma}_{ab}, \sigma_{ab}$ are constructed, between which the point-by-point difference is a quantity of the order of Δ^2, where Δ is the scale of the change in the displacement gradients defined by the relationship

$$\Delta = \sup_V (\delta^{ij} w_{k,i}^* w_{,j}^{*k})^{1/2}$$

(w^{*i} is the displacement field of points of an elastic body and the solution of the geometrically non-linear problem). In the norm L_2 the difference mentioned will be a quantity of the order of $\Delta^2 \mid V \mid^{1/2} (\mid V \mid$ is the volume of the domain V), which yields the required error estimate of the solutions of the geometrically non-linear theory in the norm L_2 for small displacement gradients because of (1.9) and (1.10).

Theorem. As compared with the geometrically non-linear theory the error in the solutions of the geometrically linear theory of elasticity is determined by the inequalities:

$$\| \sigma_{ij}^\circ - \sigma_{ij}^* \| \leqslant C_1 \Delta^2 | V |^{1/2}, \quad \| \varepsilon_{ij}^\circ - \varepsilon_{ij}^* \| \leqslant C_2 \Delta^2 | V |^{1/2} \qquad (1.11)$$
$$\| w_i^\circ - w_i^* \| \leqslant C_3 \Delta^2 | V |^{1/2}$$

where σ_{ij}^*, ε_{ij}^*, w_i^* are the solution of the geometrically non-linear problem, σ_{ij}°, ε_{ij}°, w_i° are the solution of its corresponding geometrically linear problem, and Δ is the scale of variation of the displacement gradients in the non-linear problem. The last inequality in (1.11) assumes no rigid body displacements. The constants C_1, C_2, C_3 depend only on the tensor of the elastic moduli E^{ijkl}. Moreover, the constant C_3 also depends on the geometry of the domain that the elastic body occupies in the undeformed state.

Proof. Let p_i^{*a} be the Piola-Kirchhoff tensor corresponding to the solution of the non-linear problem. For simplicity in the subsequent discussions, we set $x^{\circ i}(\xi^a) = \xi^i$ (in the undeformed state the Lagrange and Cartesian coordinates coincide). This permits identification of the indices $i, j, k \ldots$ with $a, b, c \ldots$. Then the equilibrium equations of the non-linear theory (1.4) take the form $p_{,j}^{ij} + F^i = 0$ and are in agreement with the equilibrium equations $\sigma_{,j}^{ij} + F^i = 0$ of the geometrically linear theory. It hence follows that the Piola-Kirchhoff stress field p_{ij}^* is statically admissible in the geometrically linear problem (the static boundary conditions (1.5) are common for the linear and non-linear problems).

The kinematically admissible field of stresses σ_{ij} corresponding to the displacement field w_i^* has the form $\sigma^{ij} = E^{ijkl}(w_{k,l}^* + w_{l,k}^*)/2$. Then

$$\left| x_l^{*i} \frac{\partial U}{\partial \varepsilon_{lj}} - \sigma^{ij} \right| = | w_{,l}^{*i} E^{jlmn} (w_{m,n}^* + w_{n,m}^* + \qquad (1.12)$$
$$w_{,m}^{*s} w_{s,m}^*)/2 + E^{ijmn} w_{,m}^{*s} w_{s,n}^*/2 | \leqslant \varkappa \Delta^2$$

where $\varkappa$ is independent of Δ and is just a function of the elastic constants of the material.

As statically admissible in the linear problem we select the field of stress $p^{ij} : \bar{\sigma}^{ij} = p^{ij}$. Since $E^* (\sigma_{ij}^\circ - \sigma_{ij}) \leqslant E^* (\sigma_{ij} - \bar{\sigma}_{ij})$ then by using (1.10), (1.12) as well as the condition that $p_{ij}^* = \sigma_i^{*k} (\delta_{kj} + w_{k,j}^*)$ while σ_{kl}^* is a quantity of order Δ, we obtain the first estimate of (1.11). By virtue of the linearity of Hooke's law, the error estimate of the gometrically linear theory in deformations follows from it at once (the second estimate in (1.11)).

We will show that an analogous estimate also holds for the displacement field. We will assume that the rigid body displacements are eliminated because of some constraints. We let $\bar{\bar{\varepsilon}}_{ij}$ denote the linearized strain tensor corresponding to the displacement field $w_i^* : \varepsilon_{ij} = (w_{i,j}^* + w_{j,i}^*)/2$. As a corollary of the Korn inequality

$$\| (\delta^{ij} w_{,i}^{*k} w_{k,j}^*)^{1/2} \| \leqslant K_0 \| \varepsilon_{ij} \| \qquad (1.13)$$

under the constraints made on the displacement field, the following estimate holds

$$\| w_i^* \| = \Big[\int\limits_V w^{*i} w_i^* \, d\tau \Big]^{1/2} \leqslant K \| \bar{\bar{\varepsilon}}_{ij} \| \qquad (1.14)$$

Hence

$$\| w_i^\circ - w_i^* \| \leqslant K \| \varepsilon_{ij}^\circ - \bar{\bar{\varepsilon}}_{ij} \| \qquad (1.15)$$

The constants K_0, K in (1.13), (1.14) depend only on the geometry of the domain V. Since $\varepsilon_{ij}^* = \bar{\bar{\varepsilon}}_{ij} + (w_{,i}^{*k} w_{k,j}^*)/2$ the inequality

$$\| \varepsilon_{ij}^* - \bar{\bar{\varepsilon}}_{ij} \| \leqslant {}^1/_2 K_0 \| \bar{\bar{\varepsilon}}_{ij} \|^2 \qquad (1.16)$$

follows from (1.13).

We rewrite the right side of (1.15) as follows $\| \varepsilon_{ij}^\circ - \bar{\bar{\varepsilon}}_{ij} \| \leqslant \| \bar{\bar{\varepsilon}}_{ij} - \varepsilon_{ij}^* \| + \| \varepsilon_{ij}^* - \varepsilon_{ij}^\circ \|$ which together with (1.19) and (1.11) yield the estimate required (the last inequality in (1.11)).

Remark 1°. The estimates (1.11) can be modified if the quantity

$$\Delta^* = \| (\delta^{ij} w_{,k}^{*k} w_{k,j}^*)^{1/2} \| \, | V |^{-1/2}$$

is introduced as the scale of variation of the displacement gradients.

Then the inequality (1.12) takes the form

$$\| p_{ij}^* - \sigma_{ij} \| \leqslant \varkappa \Delta^{*2} | V |^{1/2}$$

This does not alter the course of the discussion, and the modification of the inequalities

(1.11) is related to the replacement of Δ by Δ^*.

2°. The estimates (1.11) are integral estimates. Appropriate point-by-point estimates follow directly from the Sobolev imbedding theorems. The question of transferring from integral estimates of the form (1.11) to point-by-point estimates is examined in detail in /6/, for instance.

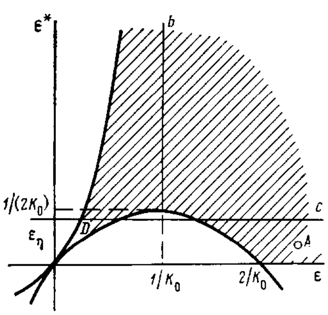
2. Linearization of geometrically non-linear problems for small deformations.

The legitimacy of linearizing geometrically non-linear problems for small displacement gradients was proved in Sect.1. It is natural to pose the question: would an analogous assertion be valid if only the deformations are small. The answer is in the negative. An illustration is thin shells; despite the smallness of the deformations their displacement can reach finite values. Let us mention certain criteria enabling a class of problems to be isolated in which the linearization is legitimate under small deformations.

Let $\varepsilon^*, \bar{\varepsilon}$ be the root mean square amplitudes of true and linear deformations $\varepsilon^* = |V|^{-1/2}\|\varepsilon_{ij}^*\|$, $\bar{\varepsilon} = |V|^{-1/2}\|\bar{\varepsilon}_{ij}\|$. The triangle inequality enables us to obtain the following relationship from (1.16)

$$|\varepsilon^* - \bar{\varepsilon}| \leqslant {}^1/_2 K_0 \bar{\varepsilon}^2 \tag{2.1}$$

We assume that the constant K_0 in inequality (1.13) is the best. It is clear that not all positive numbers $\varepsilon^*, \bar{\varepsilon}$ satisfy (2.1). The set of solutions of inequality (2.1) in the plane $\{\varepsilon^*, \bar{\varepsilon}\}$ occupies a certain domain (it is bounded by the parabolas $\varepsilon^* = {}^1/_2 K_0 \bar{\varepsilon}^2 + \bar{\varepsilon}$, $\varepsilon^* = {}^1/_2 K_0\, \bar{\varepsilon}^2 + \bar{\varepsilon}^2$ in the figure and is shaded).



We assume that the true deformations of the elastic body are small: $\varepsilon^* \ll 1$. If it follows from this that the linear deformations are also small ($\bar{\varepsilon} \ll 1$) then by virtue of (1.13) the displacement gradients will also turn out to be small ($\Delta^* \ll 1$ for $K_0 = O(1)$). It then follows from the theorem proved in Sect.1 that linearization of geometrically non-linear problems is legitimate even for small deformations. However, inequality (3.1) (see the figure) allows those states of the elastic body for which $\varepsilon^* \ll 1$ while $\bar{\varepsilon} \sim 1$ (for instance, the point A). Such a situation is obviously observed for shells. We consider it in more detail.

A point in the shaded domain in the figure corresponds to any deformable state of the elastic body, and a curve issuing from the origin corresponds to the process of deformation from the unstressed state. In order to be incident at the point A during deformation, it is necessary to intersect the vertical line b that corresponds to the states of stress with $\varepsilon^* \geqslant 1/(2K_0)$. Let ε_n denote the limiting elastic deformations in the elastic body. If $K_0 > 1/(2\varepsilon_n)$, then it follows from the above that the state A is not allowable within the framework of the model of a physically linear material. In this case the space of elastic body states in the plane $\varepsilon^*, \bar{\varepsilon}$ is bounded by the line c and is represented by the domain D. It can be shown that $\bar{\varepsilon} < 2\varepsilon^*$ for the states of stress belonging to the domain D. The latter at once results in the fact that the linearization of geometrically non-linear problems is legitimate in such a situation even for small deformations.

The Korn constant K_0, a function of the geometry of the domain V which the elastic body occupies even in the undeformed state, is in the condition $\varepsilon_n < 1/(2K_0)$. Therefore, determination of the class of problems in which the linearization is legitimate even for small deformations would reduce to investigation of the dependence of the Korn constant on the geometry of the undeformed state of an elastic body. Let us summarize what has been proved.

Lemma. Linearization of geometrically non-linear problems for small deformations is legitimate for elastic bodies for which the Korn constant K_0 is less than $1/(2\varepsilon_n)$, where ε_n are the ultimate elastic deformations. The error in the solution of the geometrically linear problem in the norm L_2 is here the magnitude of the deformations themselves.

Finding the Korn constant for any of the domains is a separate and not quite so simple mathematical problem. We will merely note that for bodies geometry has small parameters (shells, plates, and rods) the linearization of geometrically non-linear problems for small deformations can result in errors. This is related to the following. Let h be the shell or bar thickness. It is shown* (*See also: Misyura, V.A., Effect of losses of accuracy of the classical theory of shells. Candidate Dissertation, Moscow State University, 1984) that the Korn constant for such bodies grows as C/h or more rapidly as $h \to 0$ (C is independent of h). For h small K_0 becomes greater than $1/(2\varepsilon_n)$. In this case the state A becomes allowable within the framework of the model of an elastic material. Therefore, smallness of the deformation does not here ensure smallness of the displacement gradients, which indeed clarifies the possibility of error origination during linearization of non-linear problems only for small deformations.

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DYNAMIC DEFORMATION OF INCOMPRESSIBLE MEDIA*

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A class of plane and axisymmetric problems concerning incompressible media with power law hardening, deformed over time according to special laws, is considered. Such media include, in fact, hardening plastic, non-linearly elastic and non-linearly viscous bodies whose compressibility can be neglected. The dynamic effects are studied under which the points of the body execute oscillatory or monotonic motions with respect to time. The external forces corresponding to dynamic deformation of the media in question are given. Problems of unloading are omitted for brevity; only the stages of the motion leading to loading will be considered.

Wave processes in plastic and other non-linear compressible bodies have been investigated in many papers (/1-8/ et al.). The problems of dynamic deformation under the assumption that the material is incompressible merits special attention, especially from the point of view of determining how the inertial forces affect the strength of the bodies.

1. Plane deformation. The relations for the medium in question under the conditions of plane deformation are given in polar coordinates and in the usual notation in the form of: the equations of motion

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}=\rho\frac{\partial^2 u}{\partial t^2} \tag{1.1}$$
$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2}{r}\tau_{r\theta}=\rho\frac{\partial^2 v}{\partial t^2}$$

the relation connecting the deformation and stress intensities, and the relations connecting the deformation, stress and displalement components

$$\varepsilon_0=k\sigma_0{}^n \tag{1.2}$$
$$\varepsilon_0=\sqrt{(\varepsilon_r-\varepsilon_\theta)^2+4\gamma_{r\theta}^2},\quad \sigma_0=\frac{1}{2}\sqrt{(\sigma_r-\sigma_\theta)^2+4\tau_{r\theta}^2}$$

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